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ON THE L^p-BOUNDS FOR MAXIMAL FUNCTIONS ASSOCIATED TO CONVEX BODIES IN $\mathbf{R}^n$

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ABSTRACT

The dimension-free L^2-maximal inequality for convex symmetric bodies obtained in [2] is extended for $p > \frac{3}{2}$.

1. Introduction

The results presented here are a continuation of the work [7], [2], [3] on the behavior of high dimensional maximal functions. Let B be a convex symmetric body in $\mathbf{R}^n$ and define the corresponding maximal function

$$Mf(x) = M_B f(x) = \sup_{t>0} \frac{1}{\operatorname{Vol} B} \int_B |f(x+ty)|\, dy; \qquad f \in L^1_{\mathrm{loc}}(\mathbf{R}^n).$$

The main result of [2] asserts then the existence of an absolute constant D satisfying

$$(1) \qquad \| M_B f \|_{L^2(\mathbf{R}^n)} \leqq D \| f \|_{L^2(\mathbf{R}^n)}$$

which we write shortly $\| M_B \|_{2\to 2} \leqq D$. Of course, by interpolation and the obvious L^∞-bound, (1) also implies

$$(2) \qquad \| M_B \|_{p \to p} \leqq D \qquad \text{if } 2 \leqq p \leqq \infty.$$

This paper deals with the dependence of the bounds when $p < 2$. Consider the "diadic" maximal operator

$$M_1 f(x) = \sup_{j \in \mathbf{Z}} \frac{1}{\operatorname{Vol} B} \int_B |f(x + 2^j y)|\, dy.$$

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Obviously $M_1 f \leqq Mf$. The following fact is shown in the next section.

THEOREM 1. *For $1 < p < \infty$, there is a constant C_p such that*

$$\|M_1 f\|_p \leqq C_p \|f\|_p \tag{3}$$

and where C_p is independent of the body B and the dimension n.

Theorem 1 is exploited to derive

THEOREM 2. *For $p > \frac{3}{2}$, there is a constant C'_p satisfying*

$$\|Mf\|_p = \|M_B f\|_p \leqq C'_p \|f\|_p \tag{4}$$

and which is again independent of B and the dimension n.

This result brings further progress on the problems considered in [7], where these investigations were initiated.

Theorem 2 is easily deduced from Theorem 1 and the next lemma, proved in Section 3.

LEMMA 1. *If $1 < q < p \leqq 2$ and $p > 3/(1+1/q)$, then*

$$\|M\|_{p\to p} \leqq C(p,q)\|M_1\|_{q\to q}^{(1/p-1/2)/(1/q-1/2)}.$$

In the proofs of both Theorem 1 and Lemma 1, Fourier analysis will be essentially used.

Denote $(P_t)_{t>0}$ the Poisson semigroup on $\mathbf{R}^n$, thus $\hat{P}_t(\xi) = e^{-t|\xi|}$. Recall the maximal and g-function inequalities (see [4], section 2) for $1 < p < \infty$:

$$\left\| \sup_{t>0} |f * P_t| \right\|_p \leqq C(p-1)^{-1}\|f\|_p, \tag{5}$$

$$\left\| \left\{ \int_0^\infty \left| \frac{\partial P_t}{\partial t} f \right|^2 t dt \right\}^{1/2} \right\|_p \leqq \left\| \left\{ \int_0^\infty |\nabla u(x,t)|^2 t dt \right\}^{1/2} \right\|_{L^p(dx)} \tag{6}$$
$$\leqq C(p-1)^{-1}\|f\|_p$$

where $u(x,t) = (f * P_t)(x)$ and C is a constant independent of the dimension n.

REMARKS. (1) In several cases, the restriction $p > 3/(1+q^{-1})$ in Lemma 1 can be relaxed, improving on the condition $p > \frac{3}{2}$ in Theorem 2.

(2) The author has previously proved Theorem 1 for the special case of the Cartesian cubes $[-\frac{1}{2}, \frac{1}{2}]^n$. In this argument, geometric properties of the cube played a rôle.

(3) The numbers $\|M\|_{p\to p}$ and $\|M_1\|_{p\to p}$ are preserved when replacing the body

B by an affine image $v(B)$, $v \in GL(\mathbf{R}^n)$. For instance,

$$M_{v(B)}f = M_B(f \circ v) \circ (v^{-1}).$$

Hence, by the results of [2], we may and do assume that the Fourier transform of the indicator function χ_B satisfies the properties

$$|\hat{\chi}_B(\xi)| \leqq C(L \cdot |\xi|)^{-1}, \tag{7}$$

$$|1 - \hat{\chi}_B(\xi)| \leqq CL \cdot |\xi| \tag{8}$$

and

$$|\langle \nabla \hat{\chi}_B(\xi), \xi \rangle| \leqq C, \qquad \text{for all } \xi \in \mathbf{R}^n \tag{9}$$

where $L = L(B)$ is a number dependent on B. Here and in the sequel, the letter C will always stand for absolute constants.

Only properties (7), (8), (9) will be used in proving Theorems 1 and 2 (cf. remark at the end related to the limitations of this method).

2. The estimates for the diadic maximal operator

In this section, Theorem 1 will be derived. Let the body B be fixed and assume (7), (8), (9) valid. Our purpose is to prove *a priori* inequalities on the numbers $A(p,q)$ defined as the best constant fulfilling the inequality

$$\left\| \left(\sum_j |f_j * (\chi_B)_{2^j}|^q \right)^{1/q} \right\|_p \leqq A(p,q) \left\| \left(\sum |f_j|^q \right)^{1/q} \right\|_p \tag{10}$$

where in general $K_t(x) = t^{-n}K(t^{-1}x)$. Here $1 < p < \infty$ and $1 \leqq q \leqq \infty$. Theorem 1 thus consists in finding an absolute bound for $A(p) \equiv A(p,\infty)$, which is a decreasing function of p on $]1,\infty]$. Of course, we may suppose $f_j \geqq 0$ in (10). Note that by duality

$$A(p,1) = A(p',\infty) \tag{11}$$

and by the interpolation property (see [1] for instance) $L^p_{l^2} = [L^p_{l^\infty}, L^p_{l^1}]_{1/2,2}$, also

$$A(p,2) \leqq A(p,1)^{1/2} A(p,\infty)^{1/2}. \tag{12}$$

Suppose $p \leqq 2$. By (11), (12) and (2) asserting that $A(p',\infty) \leqq D$, we may write

$$A(p,2) \leqq D^{1/2} A(p,\infty)^{1/2}. \tag{13}$$

This fact will be essentially exploited in the sequel.

The aim of the following reasoning is to get a reverse inequality, estimating

$A(q)$ in terms of $A(p,2)$ for $p<q$. Define $K=\chi_B-P_L$ with L as in (7), (8). Fix a positive integer s. Then

$$M_1 f=\sup_{j\in\mathbf{Z}}|f*(\chi_B)_{2^j}| \le \Big(\sum_j|\Delta_j f*K_{2^j}|^2\Big)^{1/2}+\Big(\sum_j|g_j*K_{2^j}|^2\Big)^{1/2}+\sup_{t>0}|f*P_t| \tag{14}$$

denoting

$$\Delta_j f=(f*P_{L\cdot 2^{j-s}})-(f*P_{L\cdot 2^{j+s}})\quad\text{and}\quad g_j=f-\Delta_j f.$$

We analyse the contributions of the three terms in the right member of (14). By definition

$$\Big\|\Big(\sum_j|\Delta_j f*K_{2^j}|^2\Big)^{1/2}\Big\|_p\le A(p,2)\Big\|\Big(\sum_j|\Delta_j f|^2\Big)^{1/2}\Big\|_p + \Big\|\Big(\sum_j(P_{2^j}*\Delta_j f)^2\Big)^{1/2}\Big\|_p. \tag{15}$$

Since

$$\Delta_j f=\int_{L2^{j-s}}^{L2^{j+s}}\Big(\frac{\partial P_t}{\partial t}f\Big)dt,$$

it follows from the Cauchy–Schwartz inequality and (6) that

$$\Big(\sum|\Delta_j f|^2\Big)^{1/2}\le\Big\{\sum\Big(\int_{L2^{j-s}}^{L2^{j+s}}\Big|\frac{\partial P_t}{\partial t}f\Big|^2 t\,dt\Big)\Big(\int_{L2^{j-s}}^{L2^{j+s}}\frac{dt}{t}\Big)\Big\}^{1/2}$$

$$\le Cs\Big(\int_0^\infty\Big|\frac{\partial P_t}{\partial t}f\Big|^2 t\,dt\Big)^{1/2},$$

$$\Big\|\Big(\sum|\Delta_j f|^2\Big)^{1/2}\Big\|_p\le C(p-1)^{-1}s\,\|f\|_p.$$

A similar estimate holds for the second term in (15). Thus, denoting μ the Lebesgue measure on $\mathbf{R}^n$,

$$\mu\Big[\Big(\sum|\Delta_j f*K_{2^j}|^2\Big)^{1/2}>\lambda\Big]\le C(A(p,2)(p-1)^{-1}s\lambda^{-1}\|f\|_p)^p,\qquad\lambda>0. \tag{16}$$

By Parseval's identity

$$\Big\|\Big(\sum_j|g_j*K_{2^j}|^2\Big)^{1/2}\Big\|_2 = \Big(\sum_j\|g_j*K_{2^j}\|_2^2\Big)^{1/2}\ \Big\{\int|\hat f(\xi)|^2\Big[\sum_j|1-e^{-2^{j-s}L|\xi|}+e^{-2^{j+s}L|\xi|}|^2|\hat K(2^j\xi)|^2\Big]d\xi\Big\}^{1/2}.$$

Since, by (7), (8),

$$|\hat{K}(\xi)| \leqq C \frac{L|\xi|}{1+L^2|\xi|^2}$$

one easily checks the pointwise inequality

$$\sum_j |1-e^{-2^{j-s}L|\xi|}+e^{-2^{j+s}L|\xi|}|^2|\hat{K}(2^j\xi)|^2 \leqq C2^{-cs}$$

where $c>0$ is some constant. So

$$\left\| \left(\sum |g_j * K_{2^j}|^2 \right)^{1/2} \right\|_2 \leqq C2^{-cs}\|f\|_2,$$

$$\mu\left[\left(\sum |g_j * K_{2^j}|^2 \right)^{1/2} > \lambda \right] < C(2^{-cs}\lambda^{-1}\|f\|_2)^2. \tag{17}$$

From (5) the distributional inequality on the last term in (14) is immediate. From the estimates (16), (17) there follows

$$\mu[M_1 f > \lambda] \leqq C[A(p,2)(p-1)^{-1}s\lambda^{-1}\|f\|_p]^p + C[2^{-cs}\lambda^{-1}\|f\|_2]^2. \tag{18}$$

Since $|f*(\chi_B)_t| \leqq \|f\|_\infty$, we may replace in the right member of (18) the function f by

$$f^\lambda = f\chi_{[|f|>\lambda/2]}.$$

Notice that since $p<2$

$$(\lambda^{-1}\|f^\lambda\|_2)^2 \geqq \lambda^{-2}\left(\frac{\lambda}{2}\right)^{2-p}\int |f^\lambda|^p \geqq c(\lambda^{-1}\|f^\lambda\|_p)^p.$$

Choosing

$$s \sim \log\left(C\frac{(\lambda^{-1}\|f^\lambda\|_2)^2}{(\lambda^{-1}\|f^\lambda\|_p)^p} \right),$$

(18) becomes

$$\mu[M_1 f > \lambda] \leqq C(p-1)^{-p}A(p,2)^p\left[\log\left(C\frac{(\lambda^{-1}\|f^\lambda\|_2)^2}{(\lambda^{-1}\|f^\lambda\|_p)^p} \right) \right]^p (\lambda^{-1}\|f^\lambda\|_p)^p.$$

From the inequality

$$\log x \leqq C\tau^{-1}x^\tau \qquad (x>2, \tau>0)$$

we get for fixed $\tau>0$ using also (13)

$$\lambda^q\mu[M_1 f > \lambda] \leqq C(p-1)^{-p}\tau^{-p}A(p)^{p/2}\|f\|_2^{2p\tau}\|f\|_p^{p-\tau p^2} \tag{19}$$

where

$$q = p + p(2-p)\tau.$$

Introducing the Lorentz-norms $L^{q,1}(\mathbf{R}^n)$ and $L^{q,\infty}(\mathbf{R}^n)$, (19) means that

$$\| M_1 f \|_{q,\infty} \leqq C(p-1)^{-p/q}(q-p)^{-p/q} A(p)^{p/2q} \| f \|_{q,1}.^{\dagger}$$

Hence, applying the Marcinkiewicz interpolation theorem for suitable values of q in the previous inequality,

$$\| M_1 f \|_q \leqq C(p-1)^{-1}(q-p)^{-2} A(p)^{1/2} \| f \|_q \qquad (2 \geqq q > p > 1). \tag{20}$$

It follows from the definition of the numbers $A(q)$ and (20) that

$$A(q) \leqq C(p-1)^{-1}(q-p)^{-2} A(p)^{1/2}. \tag{21}$$

From this fact the proof of Theorem 1 will easily be completed. It is easily seen that $\sup_{1<p\leqq 2}(p-1)A(p) < \infty$ (from [7] this expression can in fact be bounded by $\log n$).

Denote T the smallest constant such that $A(p) \leqq T(p-1)^{-6}$ and choose $\bar{p}$ satisfying $A(\bar{p}) > \frac{1}{2}T(\bar{p}-1)^{-6}$. If in (21) we let $q = \bar{p}$ and $p = \frac{1}{2}(1+\bar{p})$, then

$$\frac{T}{2}(\bar{p}-1)^{-6} \leqq C(\bar{p}-1)^{-3} A(p)^{1/2} \leqq C(\bar{p}-1)^{-3} T^{1/2}(p-1)^{-3}$$

from whence

$$T \leqq C'.$$

This ends the proof of Theorem 1.

3. Proof of Lemma 1

For $r = 1, 2, \ldots$, define the auxiliary maximal operators

$$M_r f = \sup_{t \in I_r} \frac{1}{\operatorname{Vol} B} \int_B f(x+ty)dy \qquad \text{where } I_r = \{2^{j/r}; j \in \mathbf{Z}\} \quad \text{and} \quad f \geqq 0.$$

For $r = 1$, the diadic maximal operator considered above is obtained. Further

$$Mf = \lim_{r\to\infty} M_r f \leqq M_1 f + \sum_{s=0}^{\infty} | M_{2^{s+1}} f - M_{2^s} f |. \tag{22}$$

† By convexity and definition of $\| \quad \|_{q,1}$, it suffices to check this for f of the form $|A|^{-1/q}\chi_A$. But then, this is just (19).

It is easily verified that

$$(23)\qquad |M_{2r}f - M_rf| \leqq \sup_{t\in I_r}\left|\frac{1}{\operatorname{Vol} B}\int_{B^r}[f(x+t2^{1/2r}y)-f(x+ty)]dy\right|.$$

From (22) and interpolation, it follows for $1<q<p\leqq 2$, $p^{-1}=(1-\theta)q^{-1}+\theta\frac{1}{2}$,

$$(24)\qquad \|M\|_{p\to p}\leqq \|M_1\|_{p\to p}+\sum_{s=0}^{\infty}\|M'_{2^s}\|_{q\to q}^{1-\theta}\|M'_{2^s}\|_{2\to 2}^{\theta},$$

defining

$$M'_rf=\sup_{t\in I_r}|[\chi_B-(\chi_B)_{2^{1/2r}}]_t*f|.$$

Splitting $I_r=\bigcup_{\alpha=0}^{r-1}I_r^{(\alpha)}$, $I_r^{(\alpha)}=\{2^{j/r};j\in r\mathbf{Z}+\alpha\}$, it is clear that

$$\|M_rf\|_q\leqq r\|M_1\|_{q\to q}\|f\|_q$$

and therefore also

$$(25)\qquad \|M'_r\|_{q\to q}\leqq 2r\|M_1\|_{q\to q}.$$

To obtain the L^2-bound, we invoke the following majoration proved by Fourier analysis (see [2]).

LEMMA 2. *Consider a kernel $K\in L^1(\mathbf{R}^n)$ and introduce the quantities*

$$\alpha_j=\sup_{2^j\leqq|\xi|<2^{j+1}}|\hat K(\xi)|;\quad \beta_j=\sup_{2^j\leqq|\xi|<2^{j+1}}|\langle\nabla\hat K(\xi),\xi\rangle|\qquad (j\in\mathbf{Z}).$$

Then

$$(26)\qquad \left\|\sup_{t>0}|f*K_t|\right\|_2\leqq C\Gamma(K)\|f\|_2$$

where

$$K_t(x)=t^{-n}K(t^{-1}x)\quad\textit{and}\quad \Gamma(K)=\sum_{j\in\mathbf{Z}}\alpha_j^{1/2}(\alpha_j+\beta_j)^{1/2}.$$

Of course we apply this lemma for $K=K_r=\chi_B-(\chi_B)_{2^{1/2r}}$. From (7), (8) and (9), we easily get

$$|\hat K(\xi)|=|\hat\chi_B(\xi)-\hat\chi_B(2^{1/2r}\xi)|\leqq\int_1^{2^{1/2r}}|\langle\nabla\hat\chi(t\xi),\xi\rangle|dt<Cr^{-1},$$

$$|\hat K(\xi)|\leqq|\hat\chi_B(\xi)|+|\hat\chi_B(2^{1/2r}\xi)|\leqq C(|\xi|L)^{-1},$$

$$|\hat K(\xi)|\leqq|1-\hat\chi_B(\xi)|+|1-\hat\chi_B(2^{1/2r}\xi)|\leqq C|\xi|L,$$

and hence

$$\Gamma(K_r) \leqq C \sum_{j=-\infty}^{\infty} \min\left(\frac{1}{2}, \frac{L2^j}{1+L^2 4^j}\right)^{1/2} \leqq C\frac{\log r}{\sqrt{r}},$$

(27)
$$\|M'_r\|_{2\to 2} \leqq C\frac{\log r}{\sqrt{r}}.$$

Substituting (25), (27) in (24) we get

$$\|M\|_{p\to p} \leqq C\left\{\sum_{s=0}^{\infty} 2^{s(1-\theta)}\left(\frac{s^2}{2^s}\right)^{\theta/2}\right\}\|M_1\|_{q\to q}^{1-\theta} \leqq C(p,q)\|M_1\|_{q\to q}^{1-\theta}$$

provided $\theta > \frac{2}{3}$, which is the condition $p > 3/(1+1/q)$. QED

Remarks. (4) The method applied above in itself does not allow one to remove the restriction $p > \frac{3}{2}$ appearing in the statement of Theorem 2. Indeed, only properties on the kernel K,

$$K \geqq 0,$$
$$|\hat{K}(\xi)| \lesssim |\xi|^{-1}; \qquad |(1-\hat{K}(\xi))| \lesssim |\xi|,$$
$$|\langle \nabla K(\xi), \xi\rangle| < C,$$

were exploited.

If K now stands for the normalized surface measure of the 2-sphere $S^{(2)}$ in $\mathbf{R}^3$, previous conditions are fulfilled while the maximal operator

$$M_S f = \sup_{t>0} \int |f(x+ty)|\sigma(dy) \qquad (\sigma = \text{surface measure})$$

is unbounded on the space $L^{3/2}(\mathbf{R}^3)$ (see [5]).

(5) If now the conditions on K listed in (4) are replaced by

$$K \geqq 0,$$
$$|\hat{K}(\xi)| \leqq A|\xi|^{-\eta}; \qquad |1-\hat{K}(\xi)| \leqq A|\xi|,$$
$$|\langle \nabla \hat{K}(\xi), \xi\rangle| \leqq A,$$

where $\eta = 1,2,\ldots$, then more generally for $p > (\eta^2+4\eta-2)/(\eta^2+2\eta-1)$,

(28)
$$\left\|\sup_{t>0}|f * K_t|\right\|_p \leqq \phi(A,p)\|f\|_p,$$

where $\phi(A,p)$ does not depend on the dimension. The proof of (28) is an easy modification of Lemma 2 in this paper, involving the higher derivatives $\partial^{(j)}\hat{K}_t/(\partial t)^j$.

From (28), Theorem 2 may be improved in various cases. If, for instance,

$$B = B_s = [x \in \mathbf{R}^n ; \Sigma x_j^{2s} \leqq 1] \qquad (s = 1,2,\dots)$$

we get

$$\| M_B f \|_p \leqq C(p,s) \| f \|_p \qquad \text{if } p > 1$$

extending the results of [7] (section 4).

It turns out however that the condition

$$|\hat{\chi}_B(\xi)| \leqq A |\xi|^{-2}$$

(Vol $B = 1$, A bounded for increasing dimension) is already quite restrictive for the geometry of B.

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